Wavelike-particle structures in Boltzmann equation and its applications (Mathematical Analysis in Fluid and Gas Dynamics)

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Wavelike-Particle Structures in Boltzmann Equation and its Applications

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Abstract

A wavelike-particlelike structure in the Boltzmann equation was developed since 2002. This development had led to some quantitative and qualitative analysis in the nonlinear problems for the Boltzmann equation. We will give a briefly survey of this development. The dual nature property gives rise to the precise construction of the Green's function for Boltzmann equation around a global Maxwellian state. By the precise structure in Green's function, various problems such as invariant manifolds for the steady Boltzmann flows, time asymptotic nonlinear stability of Boltzmann shock layers and Boltzmann boundary layers, Riemann Problem, and bifurcation problem of boundary layer problem, etc. can be analyzed.

1 Introduction

The hard sphere collision model for the Boltzmann equation is:

$$\partial_t \mathsf{f} + \xi \cdot \nabla_{\vec{x}} \mathsf{f} = Q(\mathsf{f})/\kappa, \ \mathsf{f}(\vec{x}, t, \xi) \in \mathbb{R}, \ \vec{x}, \xi \in \mathbb{R}^3, \ \kappa > 0. \tag{1}$$

Here, $\mathsf{f}(\vec{x}, \xi, t)$ stands for the gas particle velocity density function with velocity $\xi \in \mathbb{R}^3$ at $(\vec{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$; and Q is a bilinear integral operator on the velocity density function $\mathsf{f}(x, t, \xi)$, which represents the mechanism for particle collision. One can regard the collision operator as an equilibrating mechanism. The constant $\kappa > 0$ is the Knudsen's number, which represents the mean free path of the gas flow.

This equation is a particularly interesting equation in terms of its physics nature by varying the size of κ and the sizes of the space-time scales. When $\kappa \gg 1$ and in a small space-time scales, the solution behavior resembles to free particle motions. When $\kappa \ll 1$ and space-time scales are large, the balance of the transport nature $\partial_t + \xi \cdot \nabla_x$ and the equilibrating mechanics by Q results in a conventional compressible fluid structure, which is close to the compressible Euler equation for ideal gases by the Hilbert expansion.

With the presence of a physical boundary, the gas flows behave very differently from the conventional fluid mechanics such as the thermal transpiration flows, edge flows, condensation-evaporation problems, etc. mentioned in the monograph by Sone, [35]. Grad, [9, 8, 10], also recognised an atypical nature when the presence of boundary. He proposed to have complete studies with the presences of singular layers regarding to boundary, initial data, and shock wave which are the key elements for a deep understanding of the Boltzmann equation.

Since the collision operator Q is a nonlinear integral operator, it attracts attentions of researchers to develop theories on Q such as the exponentially fast convergence to an equilibrium state for a space homogeneous problem, [2, 3]. However, those beautiful results on space homogeneous problems did not provide so much informations to study the space inhomogenous problems. The first global result on nonlinear theorem with the presence of $\xi \cdot \nabla_{\vec{x}}$ by [36] was due to a better understanding of the spectral property of the linearized Boltzmann equation $(\partial_t + \xi \cdot \nabla_x - \mathsf{L})\mathsf{g} = 0$ in [6], where L is a linear collision operator around a global Maxwellian state. The analysis on the spectrum of $-\xi \cdot \nabla_{\vec{x}} + \mathsf{L}$ is the first analytic establishment on the balance of $\xi \cdot \nabla_{\vec{x}}$ and L.

The mathematical developments on the Boltzmann equation thrilled since late '70 by various groups by different approaches and interests. Mathematically and physically, the collective behavior among $\xi \cdot \nabla_{\vec{x}}$, Q, and a physical boundary is even more interesting and complex. However, one still expects further substantial progress in this regard to achieve the understanding so that this subject is possible. On the other hand from '60 Sone [22, 23, 24, 24, 26, 27, 28, 29, 30, 31] has obtained very interesting theories regarding to boundary phenomena related to the Boltzmann equation and kinetic equations.

In year 2002 a completely different approach in the mathematical analysis for the Boltzmann equation was introduced by Liu and Yu to serve as a primary tool to undertake the analysis for the singular layers arouse from the shock layer, boundary layer, and initial layer as well as to give some partial results on Sone's discoveries. This is an approach based on the dual physical natures "wavelike-particlelike" of the Boltzmann equation. This article is aimed to review this development and its applications towards the problems by Sone and Grad.

2 Some background and motivation for Boltzmann equation and conservation laws

In [6], one considers the spectrum problem

$$(-i\xi \cdot \eta + \mathsf{L})\psi(\eta) = \sigma(\eta)\psi(\eta) \tag{2}$$

for the linear Boltzmann equation

$$\mathsf{f}_t + \xi \cdot \nabla_{\vec{x}}\mathsf{f} - \mathsf{L}\mathsf{f} = 0 \tag{3}$$

around a global Maxwellian state $\mathsf{M} = \mathsf{M}_{[1,0,\theta]}$ in the Fourier variable $\eta \in \mathbb{R}^3$, where $\mathsf{M}_{[\rho,u,\theta]}(\xi) = \rho\frac{e^{-\frac{|\xi-u|^2}{4\theta}}}{(4\pi\theta)^{3/2}}$. It is asserted that there exist $\kappa_0 > 0$ and $\kappa_1 > 0$ such that for

$|\eta| < \kappa_0$ there are five branches $\sigma_j(|\eta|) \subset \{z \in \mathbb{C} | Re(z) < 0\}$ tangential to the imaginary axis with the asymptotic for $|\eta| \ll 1$

$$\begin{cases} \sigma_1(\eta), \sigma_2(\eta) = \pm ic|\eta| - A_1|\eta|^2 + O(|\eta|^3), \\ \sigma_j(\eta) = -A_j|\eta|^2 + O(1)|\eta|^3 \text{ for } j = 3, 4, 5, \end{cases} \tag{4}$$

with $A_j > 0$, where $c = \sqrt{5\theta/3}$ is the speed of sound wave at rest; and there is a spectral gap:

$$\sigma(\eta) \notin \{Re(z) > -\kappa_1\} \text{ for } |\eta| > \kappa_0. \tag{5}$$

One can view the spectrum $\sigma(\eta)$ as a balance of the space transport mechanism $\xi \cdot \nabla_{\vec{x}}$ in the Fourier variable η and the linear collision operator L. By this spectrum property in [36], one applied a resolvent approach and a bootstrap approach to yield nonlinear stability of a global Maxwellian state M.

In [11, 21], one expanded the eigenfunction $\psi(\eta)$ in terms of the collision invariants of L so that the relationship between the Boltzmann equation and the hydrodynamic equations is clearer. The expansion of the eigenfunctions gave hints to the introduction of macro-micro decomposition in [14]:

$$\mathsf{f} = \mathsf{P}_0\mathsf{f} + \mathsf{P}_1\mathsf{f} \equiv \mathsf{f}_0 + \mathsf{f}_1, \tag{6}$$

where P_0 is a linear combination of finite number of collision invariants related to a local Maxwellian; and one can identify f_0 as a vector in $\mathbb{R}^3$ for a planar wave problem. With this decomposition, one can rewrite the time asymptotic stability for a planar wave perturbation j, $\partial_t \mathsf{j} + \xi^1 \partial_x \mathsf{j} = \frac{\delta Q}{\delta \varphi}\mathsf{j} + Q(\mathsf{j})$, of a Boltzmann shock profile φ coupled with a 3×3 viscous system through the microscopic component j_1 of j:

$$\partial_t F + A(x)F_x = B(x)F_{xx} + O(1)J(\partial_t \mathsf{j}_1), \ F \in \mathbb{R}^3. \tag{7}$$

Here, the Boltzmann shock profile φ of (1) is a travelling wave solution $\mathsf{f}(x,t) = \varphi(x - st)$ connecting two Maxwellians $\mathsf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}$ given by a hyperbolic shock wave $((\rho_-, u_-, \theta_-), (\rho_+, u_+, \theta_+))$ together with the speed s given the Rankine-Hugoniot condition.

Then, by assuming that the difference of the end states of the shock wave is sufficient small and the total macroscopic component of perturbation is zero, one shows that the Boltzmann shock profile is stable by implementing the energy method for conservation laws by [7]. The consequence of the stability is that the Boltzmann shock profile $\varphi(x, \xi)$, obtained by [1], is a positive-valued function in (x, ξ).

With the micro-macro decomposition, one can implement this energy method to work out the problem about the existence of Knudsen layers (boundary layers) with condition, $|Mach\ Number| \neq 0, 1$, [37]. The energy method was also applied to derive a macroscopic H-theorem, [16], to show the time asymptotic convergent to a hyperbolic rarefaction wave,

[18], and to show nonlinear stability of the boundary layer with Mach number less -1, [38]. When *Mach Number* > -1, the energy method can not be applied due to the fact that the solution of initial boundary value problem contains singularity at boundary so that the energy method could not be applied. It led to search for a new approach which does not require regularity property of the solution. The right candidate for such a tool is the Green's function since the Boltzmann equation is a semilinear equation.

3 Particlelike-Wavelike Duality

One starts to consider problems in planar wave solutions to establish the understanding on the natures of the Boltzmann equation, i.e. $x, \eta \in \mathbb{R}$, and $\xi \in \mathbb{R}^3$.

We start to review the work given in [15]. It begins from the consideration of the Green's function for (3). The Green's function can be represented as the inverse transform of the semigroup:

$$\mathbb{G}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta x + (-\xi^1\eta + \mathrm{L})t} d\eta. \tag{8}$$

This is an L^2_ξ-operator-valued function in (x,t), where L^2_ξ is the standard Hilbert space, $L^2(\mathbb{R}^3)$. The spectral information $\sigma(\eta)$ of (2) given in (4) poses a difficulty to obtain the Green's function for any (x,t) since there is no spectral information $\sigma(\eta)$ for all $|\eta| \geq \kappa_0$. In order to cope with the insufficient spectral information due to (5), one introduces a long wave-short wave decomposition of the Green's function

$$\begin{aligned}
&\mathbb{G}(x,t) = \mathbb{G}_L(x,t) + \mathbb{G}_S(x,t),\\
&\begin{cases}
\mathbb{G}_L(x,t) \equiv \dfrac{1}{2\pi}\displaystyle\int_{|\eta|<\varepsilon_0} e^{i\eta x + (-i\eta x + \mathrm{L})t} d\eta, \text{ for a fixed } \varepsilon_0 \in (0,\kappa_0),\\
\mathbb{G}_S(x,t) = 1 - \mathbb{G}_L(x,t).
\end{cases}
\end{aligned} \tag{9}$$

Here, $\mathbb{G}_L(x,t)$ is a long wave component of the Green's function. The spectrum information (4) is the core to build the long wave component for both the Boltzmann equation and linearized compressible Navier-Stokes equations. By complex analysis one can conclude the long wave component $\mathbb{G}_L(x,t)$ satisfies for $t \geq 1$ and $|x| < 2ct$ there exists $C_0 > 0$ such that

$$\|\mathbb{G}_L(x,t)\|_{L^2_\xi} \leq O(1)\left(\frac{e^{-\frac{(x+ct)^2}{C_0 t}} + e^{-\frac{x^2}{C_0 t}} + e^{-\frac{(x-ct)^2}{C_0 t}}}{\sqrt{t+1}}\right); \tag{10}$$

$$\|\partial_x^k \mathbb{G}_L\|_{L^2_x(L^2_\xi)} \leq O(1) \text{ for } k = 0, 1, 2, \cdots, \tag{11}$$

and one also has that

$$\|\mathbb{G}_S(x,t)\|_{L^2_x(L^2_\xi)} \leq O(1)e^{-t/C_0}, \tag{12}$$

where c is the sound speed at rest.

Though $\|\mathbb{G}_S\|_{L^2_x(L^2_\xi)}$ decays exponentially fast, it still does not assert that the $\|\mathbb{G}_S\|_{L^\infty_x(L^2_\xi)}$ decays sufficient fast for the purpose to study the full nonlinear problem with presence of boundaries or shock layers. To resolve the problem for obtaining the estimate for $\|\mathbb{G}_S\|_{L^\infty_x(L^2_\xi)}$, one needs to reconsider the problem (3) in the space-time domain instead of the transform domain, and one needs to spell out the linear collision operator L in details in order to catch the physics nature of the Boltzmann equation:

$$\mathsf{Lg}(\xi) = -\nu(\xi)\mathsf{g}(\xi) + \mathsf{Kg}(\xi),$$

$$\begin{cases} \nu(\xi) \geq \nu_0(1+|\xi|), \\ \mathsf{Kg}(\xi) \equiv \int_{\mathbb{R}} K(\xi,\xi_*)\mathsf{g}(\xi_*)d\xi_*, \\ K(\xi,\xi_*) \in C^\infty \text{ for } |\xi-\xi_*|>0. \end{cases} \tag{13}$$

After spelling L one rearranges (3) in the form of particle propagation (ODE along particle path):

$$\begin{cases} (\partial_t + \xi^1\partial_x + \nu)\mathsf{f} = \mathsf{Kf}, \\ \mathsf{f}(x,t,\xi) = \delta(x)\delta^3(\xi-\xi_*). \end{cases} \tag{14}$$

Then, one can perform the standard Picard's iteration in ODE for finite number of iterations with some cut-off in $K(\xi,\xi_*)$ in the first iteration to yield the following particlelike decomposition:

$$\begin{cases} \mathsf{f} = \mathbb{P} + \mathsf{R}, \\ \mathbb{P} \equiv \sum_{k=0}^{2l} \mathsf{f}_k. \end{cases} \tag{15}$$

Here, $\mathsf{R}(x,t)$ is the remainder term of the Picard iteration. The functions f_k and $\mathsf{R}(x,t)$ satisfy the property:

$$\begin{gathered} \mathsf{f}_0(x,t) = e^{-\nu(\xi)t}\delta(x-\xi^1 t)\delta^3(\xi-\xi_*), \\ \|\mathsf{f}_k(x,t)\|_{L^2_\xi} \leq O(1)e^{-(|x|+t)/C_0} \text{ for } k=3,\cdots,2l+1, \\ \partial_\xi^k \mathsf{f}_2(x,t,\xi) < \infty \text{ for } k=0,\cdots,2l, \\ \begin{cases} (\partial_t + \xi^1\partial_x - \mathsf{L})\mathsf{R} = \mathsf{Kf}_{2l+1}, \\ \mathsf{R}|_{t=0} \equiv 0. \end{cases} \end{gathered} \tag{16}$$

From the properties (16) and (4), one can only have property about the remainder $\mathsf{R}(x,t)$ there exists $C_0>0$

$$\|\mathsf{R}(\cdot,t)\|_{L^2_x(L^2_\xi)} \leq C_0 \text{ for } t>0. \tag{17}$$

Here, neither the two decompositions (9) nor (15) give the global structure of $\|\mathbb{G}(x,t)\|_{L^2_\xi}$ for all (x,t).

Denote

$$\mathsf{M}_l \equiv \underbrace{e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K} \underset{(x,t)}{*} e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K} \underset{(x,t)}{*} \cdots \underset{(x,t)}{*} e^{(-\xi^1\partial_x-\nu(\xi))t}\mathsf{K}}_{2l\ times} \underset{(x,t)}{*} e^{(-\xi^1\partial_x-\nu(\xi))t}. \tag{18}$$

Lemma 3.1 (Mixture Lemma [15]). *For each given $l \geq 0$ there exists $O_l > 0$ such that*

$$\|\partial_x^l \mathsf{M}_l \mathsf{g}\|_{L_x^2(L_\xi^2)} \leq O_l \left(\|\mathsf{g}\|_{L_x^2(L_\xi^2)} + \|\partial_\xi^l \mathsf{g}\|_{L_x^2(L_\xi^2)} \right) \text{ for } t \geq 0. \tag{19}$$

Here, $e^{(-\xi^1\partial_x - \nu(\xi))t}$ is a transport mechanism in the space-time domain and K is a mechanism to mix the velocity density distribution ξ at (x,t). This lemma asserts the conversion from the microscopic regularity ∂_ξ to the macroscopic regularity ∂_x with every two mixture of $e^{(-\xi^1\partial_x - \nu(\xi))t}\mathsf{K} \underset{(x,t)}{*} e^{(-\xi^1\partial_x - \nu(\xi))t}\mathsf{K}$. This lemma is about the conversion on the regularity through space convection and microscopic velocity.

3.1 Dual structures

Here, (10), (11), (12), (16), (17), and (19) are facts of simple mathematical analysis except (10) required some detailed complex analysis. By each own mathematical approach along, there is no much room to obtain the structure $\|\mathbb{G}(x,t)\|_{L_\xi^2}$. It is strikingly interesting that all those simple estimates binding together will generate the dual natures of the Boltzmann equation. By equating the two decompositions (9) and (15) together,

$$\begin{cases} \mathbb{P} - \mathbb{G}_S = \mathbb{G}_L - \mathsf{R}, \\ \|\partial_x^l(\mathbb{G}_L - \mathsf{R})\|_{L_x^2(L_\xi^2)} = O_l \text{ for } l \geq 2, \\ \|\mathbb{P} - \mathbb{G}_S\|_{L_x^2(L_\xi^2)} \leq O(1)e^{-t/C_0}. \end{cases} \tag{20}$$

The above and Poincare's inequality yield that

$$\|\mathsf{R} - \mathbb{G}_L\|_{L_x^\infty(L_\xi^2)} = \|\mathbb{P} - \mathbb{G}_S\|_{L_x^\infty(L_\xi^2)} \leq O(1)e^{-t/C_1} \text{ for some } C_1 > 0. \tag{21}$$

It concludes that the remainder term R and the long wave component $\mathbb{G}_L$ are exponentially close; and the compressible viscous fluid wave structure presented in R and the shortwave component $\mathbb{G}_S(x,t)$ are as follows.

$$\begin{aligned} \|\mathsf{R}(x,t)\|_{L_\xi^2} &\leq O(1)\left(\frac{e^{-\frac{(x+ct)^2}{C_0(t+1)}} + e^{-\frac{x^2}{C_0(t+1)}} + e^{-\frac{(x-ct)^2}{C_0(t+1)}}}{\sqrt{t+1}} \right); \\ \|\mathbb{P} - \mathbb{G}_S(x,t)\|_{L_\xi^2} &\leq O(1)e^{-t/C_1}. \end{aligned} \tag{22}$$

In particular, one can have a time lapse property for the remainder term R:

$$\|\mathsf{R}(x,t)\|_{L_\xi^2} \leq O(1)\int_0^t e^{-\tau/C_1}d\tau \left(\frac{e^{-\frac{(x+ct)^2}{C_0(t+1)}} + e^{-\frac{x^2}{C_0(t+1)}} + e^{-\frac{(x-ct)^2}{C_0(t+1)}}}{\sqrt{t+1}} \right). \tag{23}$$

This, (15), and (16) together conclude the particlelike-wavelike structure, $\mathbb{P}(x,t)$-$\mathsf{R}(x,t)$, of the linear Boltzmann equation (3).

3.2 Diagonal and off-diagonal hydrodynamic structure

With respect to the macro-micro decomposition $(\mathsf{P}_0, \mathsf{P}_1)$, the representation $\mathsf{P}_0\xi^1\mathsf{P}_0$ of the macroscopic transport ξ^1 is identical to the convection matrix of a linearized Euler equation. The convection matrix can be diagonalised in terms of the Riemann invariants E_j, $j = 1, 2, 3$,

$$\begin{aligned} &\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{E}_j = \lambda_j\mathsf{E}_j, \\ &(\mathsf{E}_j, \mathsf{E}_k) = \delta^j_k, \\ &\{\lambda_1, \lambda_2, \lambda_3\} = \{-c, 0, c\}, \end{aligned}$$

so that each Riemann invariant E_j propagates along a particular direction $dx/dt = \lambda_j$, where c is the speed of sound wave. Those Riemann invariants E_j and the Green's function $\mathbb{G}(x,t)$ satisfy for $t \geq 1$

$$(\mathsf{E}_l, \mathbb{G}_L(x,t)\mathsf{E}_k) \leq O(1)\frac{e^{-\frac{(x-\lambda_j t)^2}{C_1 t}}}{t^{(3-\delta^k_j-\delta^l_j)/2}} \text{ for } |x - \lambda_j t| < \frac{c}{2}t. \tag{24}$$

4 Application of the Green's function

After establishing the structure of the Green's functions for planar wave solutions, one had applied those structures to various nonlinear problems. We will outline the applications of the Green's function in this section.

4.1 Pointwise convergence to global Maxwellian state

In [15], one considers a small perturbation of the Boltzmann equation around a global Maxwellian in a 1-D space domain

$$\begin{cases} \mathsf{f}_t + \xi^1\partial_x\mathsf{f} = \mathsf{L}\mathsf{f} + \mathsf{M}^{-1/2}Q(\mathsf{M}^{1/2}\mathsf{f}), \\ \|\mathsf{f}(x,0)\|_{L^\infty_{\xi,\beta}} \leq O(1)\varepsilon\, e^{-|x|}, \ \beta \geq 5/2 \end{cases} \tag{25}$$

where $\|\mathsf{g}\|_{L^\infty_{\xi,\beta}}$ is defined by $\|(1+|\xi|)^\beta \mathsf{g}\|_{L^\infty_\xi}$. The Green's function and the lemmas in [13] for nonlinear waves coupling give the structures of the perturbations as follows.

$$\|\mathsf{f}(x,t)\|_{L^\infty_\beta} \leq O(1)\varepsilon\left(\sum_{j=1}^{3}\frac{e^{-\frac{(x-\lambda_j t)^2}{C_0(1+t)}}}{\sqrt{1+t}} + \psi_j(x,t) + e^{-(|x|+t)/C_0}\right), \tag{26}$$

where $\psi_j(x,t) = 1/\sqrt{(x-\lambda_j t)^2 + t}$, which is the dissipation wave given in [13].

4.2 Time asymptotic stability of an initial boundary value problem

In [19], one considers a global Maxwellian $\mathsf{M}_{[1,u,\theta]}$ with Mach number $\equiv u/\sqrt{5\theta/3} \notin \{-1, 0, 1\}$ in a half space domain with an imposed homogeneous boundary condition.

One begins with the linear Milne's problem:

$$\begin{cases} \mathsf{g}_t + \xi^1\partial_x \mathsf{g} = \mathsf{L}\mathsf{g}, \\ \mathsf{g}(0,t)|_{\xi^1>0} = 0, \\ \|\mathsf{g}(x,0)\|_{L^\infty_{\xi,3}} \le e^{-|x|}. \end{cases} \tag{27}$$

The Green's function $\mathbb{G}(x,t)$ for (3) plays a role to reduce the linear initial boundary problem into a pure boundary value problem by subtracting $\mathsf{h}(x,t) \equiv \int_0^\infty \mathbb{G}(x-y,t)\mathsf{g}(y,0)dy$ from $\mathsf{g}(x,t)$ to result in the boundary value problem:

$$\begin{cases} \partial_t \mathsf{j} + \xi^1\partial_x \mathsf{j} - \mathsf{L}\mathsf{j} = 0, \\ \mathsf{j}(0,t)|_{\xi^1>0} = -\mathsf{h}(0,t)|_{\xi^1>0}, \\ \mathsf{j}(x,0) \equiv 0, \end{cases} \tag{28}$$

where the function h satisfies $\|\mathsf{h}(0,t)\|_{L^\infty_{\xi,3}} \le O(1)\sum_{j=1}^3 \frac{e^{-\lambda_j^2 t}}{\sqrt{t+1}}$ due to the pointwise structure of $\mathbb{G}(x,t)$ and where $\{\lambda_1,\lambda_2,\lambda_3\} \equiv \{u - \sqrt{5\theta/3}, u, u+\sqrt{5\theta/3}\}$. For the problem (28) together with a boundary condition $\mathsf{h}(0,t)|_{\xi^1>0}$ with a pointwise structure, a upwind damping mechanism $\gamma\mathsf{B}_+$ was applied to introduce an auxiliary problem

$$\begin{cases} \partial_t \mathsf{j}_a + \xi^1\partial_x \mathsf{j}_a - \mathsf{L}\mathsf{j}_a = -\gamma\mathsf{B}_+\mathsf{j}_a, \\ \mathsf{j}_a(0,t)|_{\xi^1>0} = -\mathsf{h}(0,t)|_{\xi^1>0}, \\ \mathsf{j}_a(x,0) \equiv 0. \end{cases} \tag{29}$$

This problem can be solved globally by the energy method with an exponentially growing weighted function in x and t, where $0 < \gamma \ll 1$ and the damping mechanism B_+ was introduced in [37] for the construction of a boundary layer. Then, one uses $\mathsf{j}_a(0,t)$ as an approximation to the full boundary data $\mathsf{j}(0,t)$.

The diagonal-off diagonal structure (24) and Duhamel's principle are used to justify that the approximated full boundary function $\mathsf{j}_a(0,t)$ is a good approximation to $\mathsf{j}(0,t)$ so that one can form a geometric series $\sum_{k=1}^\infty \mathsf{j}_{a,k}(0,t)$ to represent the full boundary data $\mathsf{j}(0,t)$ and each term satisfies

$$\|\mathsf{j}_{a,k}(0,t)\|_{L^\infty_{\xi,3}} \le O(1)\gamma^{-1/4+k}\sum_{k=0}^\infty\sum_{j=1}^3 \frac{e^{-\lambda_j^2 t/C_0}}{\sqrt{t+1}}. \tag{30}$$

This yields the full boundary data $\mathsf{j}(0,t)$. With this data, $\mathbb{G}(x,t)$, and the first Green's identity together, one obtained the pointwise structure of the solution $\mathsf{j}(x,t)$ for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. With the precise structure of the linear problem (28), the nonlinear time-asymptotic stability follows.

Following the analysis for the nonlinear time asymptotic stability problem for a Maxwellian in half space domain, in [4] one continued to study the time asymptotic pointwise structure for a nonlinear problem around a Knudsen layer. The time asymptotically nonlinear stability problem for a Knudsen layer for the cases *Mach number* $\notin \{-1,0,1\}$ were concluded, and the motivation to introduce the Green's function to study the Knudsen layer was justified in this work.

4.3 Bifurcation of boundary layers

In [20], one started to analyse the Knudsen layer when the Mach number close to 0 and ± 1. The Knudsen layers constructed in [37] are under a condition that the Mach number at the far field does include ± 1 and 0. Indeed, when the Mach numbers are around 0 or ± 1, the physical behaviours of the solutions are rather singular as pointed out by Sone's works listed the reference. The Knudsen layer problem with Mach number near $\{\pm 1, 0\}$ is a bifurcation problem,

$$\begin{cases} -\xi^1 \partial_x \mathsf{F} - Q(\mathsf{F}) = 0 \text{ for } x \in \mathbb{R}_+, \\ \lim\limits_{x\to\infty} \mathsf{F}(x) = \mathsf{M}_{[\rho,u,\theta]}, \\ \mathsf{F}(0,t)|_{\xi^1>0} : \text{ posed}, \end{cases} \tag{31}$$

with respect to parameters given by the macroscopic variables of the Maxwellian $\mathsf{M}_{[\rho,u,\theta]}$ at the far field. This is a singular problem due to two facts that the system (31) is an infinite dimensional dynamical system and it also possesses a transonic behavior with Mach number close to ± 1 and a condensation-evaporation nature with Mach number is close to 0. This problem was not ready during the work in [37]. At that time the analytical tools (energy estimates) available were too primitive and too rough to realize the rich natures of the problem. The pointwise structure of the Green's function in (24) and the particlelike structure $\mathbb{P}$ given in (15) play an essential role to perform a finite dimensional reduction for the dynamical system (31). To devise a finite dimensional reduction, one will need to construct invariant manifolds for the system (31). One establishes the invariant manifolds from building concrete projection operators $\mathbb{S}_x$, $\mathbb{U}_x$, and $\mathbb{C}_0$ on $L^\infty_{\xi,3}$ for a linear system,

$$\xi^1 \partial_x \mathsf{f} - \mathsf{L}\mathsf{f} = 0, \tag{32}$$

i.e. for any $\mathsf{b} \in L^\infty_{\xi,3}$ the functions $\mathbb{S}_x\mathsf{b}$ and $\mathbb{U}_x\mathsf{b}$ give the solutions of (32) so that

$$\lim_{x\to\infty} \mathbb{S}_x \mathsf{b} = 0, \tag{33}$$

$$\lim_{x\to-\infty} \mathbb{U}_x \mathsf{b} = 0, \tag{34}$$

$$\mathbb{C}_0 \mathsf{b} \in \mathit{Range}(\mathsf{P}_0), \tag{35}$$

$$\mathsf{b} = \lim_{x\to 0+} \mathbb{S}_x \mathsf{b} + \lim_{x\to 0-} \mathbb{U}_x \mathsf{b} + \mathbb{C}_0 \mathsf{b}. \tag{36}$$

With the pointwise structure (24), one can show that the functions $\mathbb{S}_x\mathsf{b}$ and $\mathbb{U}_x$ are

$$\begin{cases} \mathbb{S}_x\mathsf{b} \equiv \int_0^\infty \mathbb{G}(x,s)\xi^1(1-\tilde{\mathsf{B}}_+)\mathsf{b}ds \text{ for } x>0, \\ \mathbb{S}_{0+}\mathsf{b} \equiv \lim_{x\to 0+} \mathbb{S}_x\mathsf{b}, \\ \mathbb{U}_x\mathsf{b} \equiv -\int_0^\infty \mathbb{G}(x,s)\xi^1(1-\tilde{\mathsf{B}}_-)\mathsf{b}ds \text{ for } x<0, \\ \mathbb{U}_{0-}\mathsf{b} \equiv \lim_{x\to 0-} \mathbb{U}_x\mathsf{b}, \\ \mathbb{C}_0\mathsf{b} = \tilde{\mathsf{P}}_0\mathsf{b}, \end{cases} \tag{37}$$

$$\begin{cases} \tilde{\mathsf{P}}_0 \equiv \sum_{k=1}^{3} \tilde{\mathsf{B}}_k, \\ \tilde{\mathsf{B}}_k\mathsf{g} \equiv \dfrac{(\mathsf{E}_k,\xi^1\mathsf{g})\mathsf{E}_k}{\lambda_k}, \\ \tilde{\mathsf{B}}_\pm \equiv \sum_{\pm\lambda_k>0} \tilde{\mathsf{B}}_k, \end{cases}$$

where $\tilde{\mathsf{P}}_0$, $\tilde{\mathsf{B}}_+$, and $\tilde{\mathsf{B}}_-$ are the Euler flux projection, the upwind Euler flux projection, and downwind Euler flux projection.

The properties (33) and (34) are due to (24). The identity (36) is due to the δ-functions in $\mathbb{P}$ (the particlelike wave) to yield a version of Gauss lemma given in Lemma 3 in [20]. Then, one has obtained the projection operators $\mathbb{S}_{0+}$, $\mathbb{U}_{0-}$, and $\mathbb{C}_0$ to the linear stable manifold, linear unstable manifold, and the linear center manifold; and one also has an exponentially decaying structures in $\mathbb{S}_x$ and $\mathbb{U}_x$ of the linear stable flows and linear unstable flows. Thus, with the exponentially decaying structures one can apply the standard construction to obtain the local stable, local center-stable manifold for (31).

When the Mach number is close to 0, and ± 1, one needs to compare the structures of the linear stable and linear unstable manifold. When the Mach number is -1, there is a 1-dimensional degeneracy to the center manifold either from the linear stable manifold or linear unstable manifold. One can calculate this degenerated vector and use it to modify the upwind damping $\tilde{\mathsf{B}}_+$ and the projection operator $\mathbb{S}_x$ into

$$\begin{cases} \mathsf{B}_3^{\sharp,\epsilon}\mathsf{g} & \equiv \dfrac{(\xi^1\mathsf{E}_3^\epsilon,\mathsf{g})}{(\xi^1\mathsf{E}_3^\epsilon,\ell_3^\epsilon)}\ell_3^\epsilon, \\ \mathbb{S}_x^{\sharp,\epsilon}\mathsf{g} & \equiv \int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1-\mathsf{B}_3^{\sharp,\epsilon})\mathsf{g}]d\tau \end{cases} \tag{38}$$

so that one can verify the continuity of the microscopic component,

$$\mathsf{P}_1 \int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1-\mathsf{B}_3^{\sharp,\epsilon})\mathsf{g}]d\tau, \tag{39}$$

where ϵ is the difference of the Mach number and -1. Then, by energy estimates one can have the uniformly exponentially decaying structure in x when $\epsilon > 0$ and together with an algebraic condition (148) in [20] on the macroscopic and microscopic component to yield the uniformly exponentially decaying upper bound $e^{-\alpha x}$ for $x > 0$ of $\|\mathbb{S}_x^{\sharp,\epsilon}\|_{L^2_\xi}$ and with the uniform structure in $\epsilon > 0$. By taking the limits $\epsilon \to 0+$, it follows

$$\begin{cases} \mathsf{b} = \mathring{\mathbb{S}}_{0+}\mathsf{b} \oplus \mathring{\mathbb{C}}_0\mathsf{b} \oplus \mathring{\mathbb{U}}_{0-}\mathsf{b}, \\ dim(Range(\mathring{\mathbb{C}}_0)) = 4, \end{cases} \tag{40}$$

where $\mathring{\mathbb{S}}_{0+}$, $\mathring{\mathbb{U}}_{0-}$, and $\mathring{\mathbb{C}}_0$ are linear stable manifold and linear unstable manifold, and the linear center manifold. With the uniformly exponential decaying upper bound of $\|\mathring{\mathbb{S}}_x\|_{L^2_\xi}$ for $x > 0$, one can construct the local centre-unstable manifold. By taking the limit of $\epsilon \to 0-$, then one can construct the local unstable and center-stable manifolds; and the dimension of the nonlinear center manifold is 4. Since all Maxwellian states M are equilibrium states of the dynamical system, they are all in the center manifold. Due to the fact that the collision operator is orthogonal to the collision invariant, the macroscopic flux $\vec{q} = \mathsf{P}_0\xi^1\mathsf{M}$ is an invariant 3-vector of the dynamical system. This gives a three constraints to the 4-dimensional center manifold and yields a 1-dimensional invariant manifold in the center manifold with two fixed points corresponding to the Maxwellians $(\mathsf{M}_-^{\vec{q}}, \mathsf{M}_+^{\vec{q}})$, which are related to the end states of a shock wave. Then, by using the coordinate of the linear center manifold and linear stable manifold one can obtain a two scale dynamical system in the center-stable manifold with two co-dimension 2 invariant manifolds at the equilibrium states $\mathsf{M}_-^{\vec{q}}$ and $\mathsf{M}_+^{\vec{q}}$. The flows on the two co-dimension 2 will converge to the equilibrium state with an uniform exponential rate. Otherwise, it behaviours like a Burgers' equation (compressible fluid like). We illustrate the phase diagram of the center-stable manifold of the dynamical system given by (31) around a Maxwellian M_0 state with Mach number$= -1$.

The dynamical system on the 1-D invariant curve (center manifold) is a Burgers type ODE (First order ODE). This flow concludes a connecting orbit for the two states $(\mathsf{M}_-^{\vec{q}}, \mathsf{M}_+^{\vec{q}})$. This proves the existence of Boltzmann profile as well as the monotone property of the profile. This monotone property is a problem raised in [14]. Here, the two co-dimension 2 invariant submanifolds of the center-stable manifold define two scalar functions K_- and K_+ on the center-stable manifold so that the function K_- gives the bifurcation of the dynamical system; and the function K_+ defines the hydrodynamics flows patterns, either a slowly expanded pattern for flows in the region $K_+ < 0$ or an exponentially fast compressive wave pattern in the region $\{K_+ > 0\} \cup \{K_- < 0\}$. With these two functions, one can return to the bifurcation of the Milne's problem (31). By Lemma 20 in [20], there is a local 1-1 continuous map $\iota_{\vec{q}}$ from the center-stable manifold with given macroscopic flux $\vec{q}$ to the space $L^\infty_{\xi,3,+}$, which is the space for the imposed boundary data. Thus, the sign of the function $K_-(\iota_{\vec{q}}(\mathsf{b}))$ gives the bifurcation of the Milne's problem around the Mach number $= -1$. When Mach number is around 0, the result in [20] gives the Sone's bifurcation from condensation to evaporation.

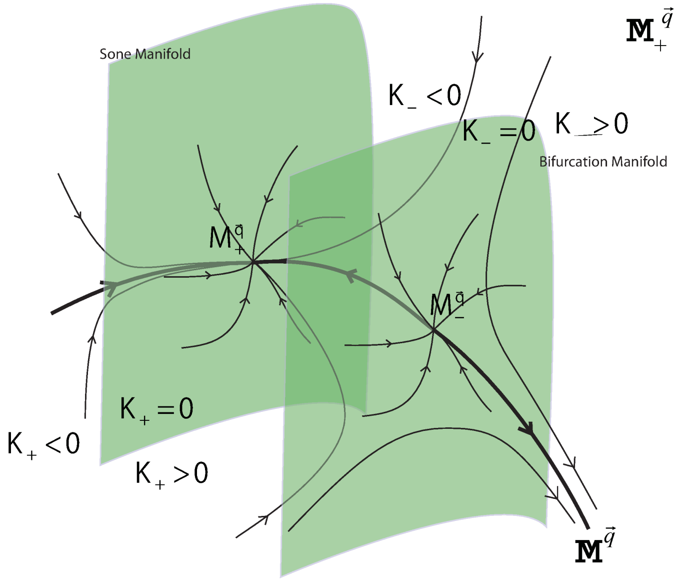


Figure 1: Two-scale dynamics on the center-stable manifold $\mathbb{M}_+^{\vec{q}}$ which is the center-stable manifold with macroscopic flux $\vec{\boldsymbol{q}} \equiv \mathsf{P}_0\xi^1\mathsf{M}_-$.

4.4 Linear and nonlinear wave scattering around a Boltzmann shock layer

In [39], one considers the Boltzmann equation around a Boltzmann shock profile, $\varphi(x-st)$:

$$\begin{cases} (\partial_t - s\partial_x \mathsf{F}) + \xi^1\partial_x\mathsf{F} - \mathsf{L}_\varphi\mathsf{F} = Q(\mathsf{F}), \\ \mathsf{F}(x,0) = \mathsf{F}_0(x), \text{ (posed initial data,)} \end{cases} \tag{41}$$

where L_φ is a linear collision operator around the shock profile φ. Suppose that the Boltzmann shock profile φ is for a weak 3-shock wave $(\vec{\boldsymbol{u}}_-, \vec{\boldsymbol{u}}_+)$ for a compressible Euler equation as a system of hyperbolic conservation laws:

$$\vec{\boldsymbol{u}}_t + \vec{\mathsf{F}}(\vec{\boldsymbol{u}})_x = 0, \ \vec{\boldsymbol{u}} \in \mathbb{R}^3.$$

One wants to remove the zero total macroscopic mass condition in [14],

$$\int_{\mathbb{R}} \mathsf{P}_0\mathsf{F}_0(x,0)dx = 0 \tag{42}$$

for the purpose to investigate the hydrodynamic limits problem for the Boltzmann equation, [9, 10].

The main point is on obtaining the optimal linear wave propagation around the Boltzmann shock layer and to use it to establish the nonlinear wave coupling. The central idea is due to viscous conservation laws. The approach to obtain the linear wave scattering around a shock profile is called the T-C scheme (transverse-compressible scheme). This scheme is closely related to the Lax's entropy condition for a p-th shock wave and the diffuse waves introduced in [12] to determine the viscous shock profile phase shift. In [39] one uses the Green's functions at two far fields to construct an approximated solution $\mathsf{A}_0(x,t)$ and a local wave front $l_0(t)\varphi'(x)$ to approximate the solution of the linearized problem

$$(\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)\mathsf{f} = 0 \tag{43}$$

to yield that $\mathscr{E}_0$, the truncation error for (43),

$$\mathscr{E}_0 \equiv (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)(\mathsf{A}_0(x,t) + l_0(t)\varphi'(x)) \tag{44}$$

satisfies that following property:

$$\begin{cases} \displaystyle\int_{\mathbb{R}} (\mathsf{D}_i, \mathscr{E}_0(x,t))dx = 0, \ i = 1, 2, \\ \|\mathsf{P}_0\mathscr{E}_0(x,t)\|_{L^\infty_{\xi,3}} \leq O(1)\dfrac{\epsilon^2}{t} e^{-(\epsilon|x|+\epsilon^2 t)/C_0} \text{ for } t \geq \epsilon^{-2}, \\ \|\mathsf{P}_1\mathscr{E}_0(x,t)\|_{L^\infty_{\xi,3}} \leq O(1)\dfrac{\epsilon}{\sqrt{t}} e^{-(\epsilon|x|+\epsilon^2 t)/C_0} \text{ for } t \geq \epsilon^{-2}, \end{cases} \tag{45}$$

where $\epsilon \equiv \|\vec{\boldsymbol{u}}_- - \vec{\boldsymbol{u}}_+\|$ and $\{\mathsf{D}_1, \mathsf{D}_2, \mathsf{M}_- - \mathsf{M}_+\}$ are the macroscopic dual vectors of $\{r_1(\vec{\boldsymbol{u}}_-), r_2(\vec{\boldsymbol{u}}_-), \vec{\boldsymbol{u}}_- - \vec{\boldsymbol{u}}_+\}$, and $r_j(\vec{\boldsymbol{u}}_-)$ are the j-th left eigenvectors of $\vec{\mathsf{F}}'(\vec{\boldsymbol{u}}_-)$. The approximated solution $\mathsf{A}_0 + l_0\varphi'$ for (43) with the property (45) is the T part of the T-C scheme.

Next, one needs to have an exponentially sharp estimate of the output $w(x,t)$ due to the truncation error $\mathscr{E}(x,t)$:

$$\begin{cases} (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)w = -\mathscr{E}_0, \\ \displaystyle\int_{\mathbb{R}} \mathsf{P}_0 w(x,0)dx = 0. \end{cases} \tag{46}$$

This is a system of equations and there is no spectrum gap property to assure an exponential decaying structure though $w(0,t)$ will exponentially converge in time. For the purpose to assert an exponential estimate, one introduced a damping to the system (46):

$$\begin{cases} (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)W_0 = -\mathscr{E}_0 - \gamma \displaystyle\sum_{j=1} (\mathsf{D}_j, W_0)\mathsf{D}_j, \\ \displaystyle\int_{\mathbb{R}} \mathsf{P}_0 W_0(x,0)dx = 0, \end{cases} \tag{47}$$

with a small $\gamma > 0$. This system possesses conservation laws:

$$\int_{\mathbb{R}} \mathsf{P}_0 W_0(x,t)dx = 0, \tag{48}$$

so that with $\gamma > 0$, (48), and energy estimates one shows that this system will decay in time exponentially.

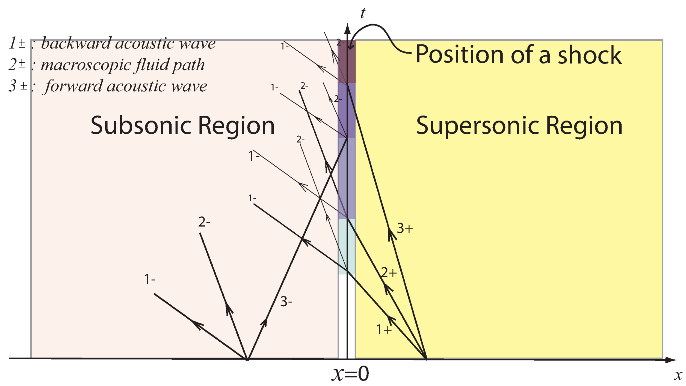
Since the truncation error $\mathsf{P}_0\mathscr{E}_0$ does not possess any transient components, the damping $-\gamma\sum_{j=1}(\mathsf{D}_j, W_0)\mathsf{D}_j$ is essentially virtual. Hence, the solution $W_0(x,t)$ gives an exponentially sharp approximation to the solution $w_0(x,t)$ around $x = 0$. The construction of the approximated solution $W_0(x,t)$ is called the C-part of the T-C scheme. This part creates another truncation error $-\gamma\sum_{l=1}^{2}(\mathsf{D}_l, W_0)\mathsf{D}_l$. Then, this leads to consider the problem

$$\begin{cases} (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)\mathsf{f}_1 = \gamma\displaystyle\sum_{l=1}^{2}(\mathsf{D}_l, W_0)\mathsf{D}_l, \\ \mathsf{f}_1(x,0) = 0. \end{cases} \tag{49}$$

One repeats the same procedure to give the T-C iteration:
To find A_i and $l_i(t)$ satisfying

$$\begin{cases} \mathscr{E}_i(x,t) \equiv (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)(\mathsf{A}_i + l_i(t)\varphi') - \gamma\displaystyle\sum_{l=1}^{2}(\mathsf{D}_l, W_{i-1})\mathsf{D}_l, \\ (\partial_t + (\xi^1 - s)\partial_x - \mathsf{L}_\varphi)W_i = -\mathscr{E}_i - \gamma\displaystyle\sum_{j=1}(\mathsf{D}_j, W_i)\mathsf{D}_j, \\ W_i(x,0) = 0, \end{cases} \tag{50}$$

and the property (45) for $\mathscr{E}_0$ still holds for $\mathscr{E}_i$. Finally, one obtained sharp linear wave scattering structure around the shock profile. The linear wave scattering structure is used to show the pointwise structure of solution of (41) as illustrated:



This T-C scheme also works for viscous conservation law. Especially, the sharp pointwise structure gives advantages in the study of the case with presence of boundary in [5].

4.5 Riemann problem for shock wave data

In [40], one considers the initial value problem (41) with a shock wave initial data $\mathsf{F}_0(x)$:

$$\mathsf{F}_0(x) = \begin{cases} \mathsf{M}_{\vec{u}_-} \text{ for } x < 0, \\ \mathsf{M}_{\vec{u}_+} \text{ for } x > 0. \end{cases} \tag{51}$$

Here, $(\vec{u}_-, \vec{u}_+)$ is a shock wave and $\mathsf{M}_{\vec{u}_\pm}$ are Maxwellians related to the states $\vec{u}_\pm$; and $\|\vec{u}_- - \vec{u}_+\| = \varepsilon \ll 1$.
This problem is a multi-time scale problem. There are five time scales illustrated by the table:

Primary wave	Valid time domain	Scale	Slip
$f(x/t)$	$0 < t < 1$	Hyperbolic scale	Initial Layer
$O(1)$	$t \sim 1$	$O(1)$ scale	Overlapping layer (a)
$\sum_{j=1}^{3} f_j(\frac{x-\lambda_j t}{\sqrt{t}})$	$1 \leq t < \varepsilon^{-2}$	Parabolic scale	Overlapping layer (b)
$v(x,t)$	$\varepsilon^{-2} \leq t \leq \lvert\log\varepsilon\rvert\varepsilon^{-2}$	Nonlinearity formation scale	Overlapping layer (c)
$\mathsf{P}_0\varphi(x)$	$t < \lvert\log\varepsilon\rvert\varepsilon^{-2}$	Time-asymptotic stability scale	Shock layer

(52)

In the time scale $0 < t < 1$, the particlelike structure $\mathbb{P}$ of the Green's function and the shock wave initial data force the solution $\mathsf{F}(x,t)$ to behave close to the hyperbolic scale function $f(x/t)$. In the time scale $t \sim 1$, one breaks the collision operator into gain and loss to yield the $O(1)$ structure. When $t \in (1, \varepsilon^{-2})$, one can linearize the problem at the Maxwellian $\mathsf{M}_{\vec{u}_-}$ or $\mathsf{M}_{\vec{u}_+}$, then by the structure (24) one concludes that the structures resemble to the convected heat equation with speeds λ_j. When $t \in (\varepsilon^{-2}, \varepsilon^{-2}\log\varepsilon)$, one restricts the macroscopic state on the line segment connecting $\mathsf{M}_{\vec{u}_-}$ and $\mathsf{M}_{\vec{u}_+}$ to form an approximated solution. This restriction carries the spirit of the Chapman-Enskog expansion. One can derive a nonlinear scalar equation close to the viscous Burgers equation. One can use the Hopf-Cole transform effectively to realize the formation of the nonlinear layer. When $t \sim \varepsilon^{-2}\log\varepsilon$, one can use the formed profile by the Burgers-like equation and compare it with the Boltzmann shock profile so that one applies the stability of a shock profile in [40] to yield the global structure of the Riemann problem with a shock wave initial data.

4.6 Future developments

The works done in [4, 15, 19, 20, 39, 40] are for planar wave motions of the Boltzmann equation. When the perturbations are multi-D, the mathematical analysis of the related problems are completely open. Indeed, there are many open problems in physics mentioned in the classical book [35].

About the Boltzmann equation in multi-D, the work in [17] gave the Green's function in 3-D space domain; and gave a wave structure related to Huygen's principle for the 3-D d'Alembert wave equation. In this aspect, it is interesting to consider the shock profile stability under a 3-D perturbations and in particular the multi-D hyperbolic scale waves interact with the viscous shock front. It is also interesting to consider the Riemann problem without assuming the shock wave data. The thermal transpiration flow derived in [35] is an interesting physical phenomenon to distinguish the difference between Boltzmann equation and conventional fluid mechanics. To investigate the geometric effects due to a physical boundary and to relate it with the geometric theory of diffractions would be very interesting as well.

It is also very interesting to complete the Grad's and Sone's program to study the interactions of the singular layers (shock layer, initial layer, and boundary layer) for 1-D problem.

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